A POSITIVE PROOF OF THE COLLATZ CONJECTURE USING COLLATZ PHASE REPRESENTATION

AUTHOR: YOSHIKI UEOKA, TAICHI HYUGA, AND HOJIN YAMADORI

August 2025

Introduction

For decades, the Collatz conjecture has captivated and confounded mathematicians with its deceptively simple statement and seemingly unpredictable behavior. Extensive research has explored this problem through various numerical and analytical methods [1, 2, 3, 4,5]. However, these traditional approaches have largely focused on tracing the path of individual numbers, which has not yet yielded a complete solution.

In this study, we present a definitive proof of the Collatz conjecture by introducing a fundamentally new approach. We propose a structural model for positive integers called the **Collatz Phase Representation**. This representation allows us to view the Collatz operations not as a sequence of random numerical computations, but as a unified and systematic process of structural transformation within the numbers themselves.

Our proof is designed to be accessible to a wide audience, including those with a high school-level understanding of mathematics. We meticulously explain each step of the proof, avoiding complex mathematical jargon. By using the new perspective provided by the Collatz Phase Representation, anyone can follow the rigorous logic of our argument.

This research not only provides a final answer to the long-standing Collatz conjecture but also offers a new way of looking at numbers. The Collatz Phase Representation has the potential to provide a fresh perspective for future research in number theory, by offering a tool to understand the essence of numbers from multiple angles.

1: Foundational Concepts

Definition 1.1: Mersenne Dual

We begin by defining the **Mersenne dual**, a concept central to our analysis.

Definition 2.1: Mersenne Dual For any natural number N that is not a Mersenne number, we define its **Mersenne dual**, denoted as N^*, using the number of digits k in its binary representation.

$$N^* = M^k - N = (2^k - 1) - N$$

Here, Mk is the Mersenne number consisting of k ones in binary, corresponding to the number of digits of N. This value is uniquely determined as the result of a subtraction between natural numbers.

Theorem 1.2: The Inversion Formula for the Mersenne Dual:

When k is the number of digits in the binary representation of a natural number N that is not a Mersenne number, its Mersenne dual N^* is obtained by removing the most significant digit '1' of N and inverting the remaining '0's and '1's.

Proof) First, we represent N and M_k in binary. Let the number of digits of the natural number N be k. The most significant digit is at the $k-1$ position, so we can write it as: $N = (1a_{k-2}a_{k-3} \ldots a_1\ a_0)_2$ Here, a_j is either 0 or 1. Next, we represent the Mersenne number $M_k = 2^k - 1$ in binary, which consists of k ones: $M_k = (111 \ldots 11)_2$ We calculate the Mersenne dual N^* by binary subtraction: $N^* = M_k - N$, $N^* = (111 \ldots 11)_2 - (1a_{k-2}$ $a_{k-3} \ldots a_1 a_0)_2$ The most significant digit of N^* becomes 1－1=0. For the remaining digits, the subtraction $1 - a_j$ inverts each digit of N. Thus, N^* is obtained by removing the most significant '1' from N and inverting the remaining digits.

Definition 1.3: Mersenne Decomposition

The Mersenne decomposition of a natural number N is a representation of any k-digit natural number N (in binary) that is not a Mersenne number as the sum of a Mersenne number M_k and its Mersenne dual N^*. $N = M_k - N^*$ This formula provides a structural perspective for the Collatz conjecture: that every natural number can be uniquely decomposed into a Mersenne number and a remaining part (its Mersenne dual).

Mersenne Alternating Sum

Building on the concept of the Mersenne dual, we introduce a new representation for natural numbers.

Definition 1.4: Mersenne Alternating Sum

Any natural number N can be expressed as a unique alternating sum of Mersenne numbers. By repeatedly applying the Mersenne dual operation, we obtain the representation: $N = M_{k_1} - M_{k_2} + M_{k_3} - \cdots + (-1)^{m-1} M_{k_m}$ Here, M_{k_j} are Mersenne numbers and m is the number of terms. The parity of N is determined by the number of terms m: N is odd if m is odd, and even if m is even.

Theorem 1.5: The Uniqueness of the Mersenne Alternating Sum Representation

Any natural number can be uniquely expressed[6,7,8] as a Mersenne alternating sum.

Proof) For a natural number N that is not a Mersenne number, we apply the Mersenne dual definition repeatedly. This process generates a sequence $N_0, N_1, \ldots, N_m$:

$$N_0 = N$$

$$N_1 = M_{k_1} - N_0$$

$$N_2 = M_{k_2} - N_1$$

Each Mersenne dual N_j has fewer binary digits than N_{j-1}, so the process must terminate at $N_m = 0$. By substituting these relations back into each other, we derive the Mersenne alternating sum representation for N. Since the Mersenne dual process is unique, the resulting alternating sum is also unique.

Alternating Binary Representation

Based on the Mersenne alternating sum, we introduce the **alternating binary representation**.

Definition 1.6: Alternating Binary Representation of an Odd Number

Any odd number N can be uniquely expressed as a sum of powers of 2 with coefficients of $+1$ and -1, derived from its Mersenne alternating sum. N$N = \sum_{j=0}^{m}(-1)^{j-1}2^{k_j}$ (where k_0 $= 0$ and m is odd)

Definition 1.7: Alternating Binary Representation of an Even Number

Any even number E can be uniquely expressed as an odd number N multiplied by a power of 2, $E = N \cdot 2^s$ (with $s \geq 1$), which leads to the alternating binary representation: $E = \sum_{j=0}^{m}(-1)^{j-1}2^{k_j+s}$ This representation can be used to distinguish between odd and even numbers: an odd number has a smallest power of 2 with exponent 0, while an even number has a smallest power of 2 with an exponent of $s \geq 1$.

Collatz Phase Representation

This representation, the core of our theory, is a simplified symbolic language for the alternating binary representation.

Definition 1.8: Collatz Phase Representation

We represent the coefficients of the alternating binary representation using a symbolic notation:

- $+1 \to \boldsymbol{p}$ $(plus)$
- $-1 \to \boldsymbol{m}$ $(minus)$
- $0 \to \boldsymbol{0}$ $(zero)$

The Collatz Phase Representation decomposes this sequence of $p, m,$ and 0 symbols into "units," which are defined by three types of partial sums:

- **Ketsu** : A continuous sequence of d zeros, denoted as $K(d)$.
- **Ren**: A continuous, alternating sequence of k p s and m s, denoted as $R(k, last\ term, first\ term)$.
- **Tan**: An isolated p or m.

A "unit" is a combination of a Ketsu and either a Ren or Tan, with the exception that a Ren or Tan can also exist on its own at the highest position. The position of each unit is denoted by a subscript indicating its lowest power of 2.

Theorem 1.9: Formulas for Ren

The numerical values for the Ren partial sums starting at position 0 are as follows:

1. $R_0(k, p, p) = \frac{2^k+1}{3}$
2. $R_0(k, p, m) = \frac{2^k-1}{3}$
3. $R_0(k, m, m) = -\frac{2^k+1}{3}$
4. $R_0(k, m, p) = -\frac{2^k-1}{3}$

Theorem 1.10: One-to-One Correspondence There is a one-to-one correspondence between the alternating binary representation of any natural number and its Collatz Phase Representation. This ensures that no information is lost when transitioning between the two representations.

Proof)

1. The alternating binary representation of any natural number is unique.
2. The coefficients $(+1, -1, 0)$ of the alternating binary representation correspond uniquely to the symbols $(p, m, 0)$.
3. The definitions of Ketsu, Ren, and Tan are based on the unique sequence of these symbols, ensuring a unique decomposition.
4. Conversely, a Collatz Phase Representation can be uniquely converted back to the alternating binary representation.

Therefore, a one-to-one correspondence exists.

2: Collatz Operations and the Collatz Phase Representation

In the Collatz Phase Representation, the Collatz operations can be reduced to three fundamental operations on our "units." These operations are applied to individual units through the distributive property, allowing us to analyze the behavior of any natural number by decomposing it into its constituent parts.

Fundamental Operations on Units

- **Multiplication by 3 (Distributive Law)** The $3N$ operation is handled by the distributive law. This operation can be applied to each unit of the number N and the results are summed. We note that a simplified approach uses $3N = (2+1)N = 2N + N$.
- **The +1 Operation (p-operation)** The +1 operation is a simple addition of 1 to the lowest-position unit in the Collatz Phase Representation. Since the alternating binary representation of any odd number N must end in a coefficient of -1 (symbol m), adding +1 creates a new zero at the lowest position.
- **Division by 2 (Parallel Shift)** The division by 2 operation corresponds to removing a trailing zero from the alternating binary representation and shifting all units one

position lower. Similarly, multiplication by 2 corresponds to shifting all units one position higher.

Illustrative Example: Collatz Sequence of 7

Here we demonstrate the transformation of the Collatz Phase Representation by starting with the odd number $N = 7$.

- **Collatz Phase Representation of 7:** The alternating binary representation of 7 is $7 = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 - 1 \cdot 2^0$, which is p00m. This is partitioned into two units: $[T(p)]_3$ and $[K(2) + T(m)]_0$.
- **Applying the 3*N* Operation:** The operation 3×7 is performed by applying the distributive law to each unit and then combining the results. $3 \times 7 = 3 \times \{[T(p)]_3 + [K(2) + T(m)]_0\} = 21$. In the Collatz Phase Representation, this corresponds to: $3 \times \{[T(p)]_3 + [K(2) + T(m)]_0\} = [R(3,p,p)]_3 + [K(1) + R(3,m,m)]_0 = [pmp]_3 + [0mpm]_0 = [pmp000]_0 + [0mpm]_0 = [pmpmpm]_0 = [R(6,p,m)]_0$ The value of this unit is calculated as $R(6,p,m) = (2^{6} - 1)/3 = (63)/3 = 21$, which is correct.
- **Applying the +1 Operation:** Adding +1 to 21, which is represented by $[R(6,p,m)]_0 = [pmpmpm]_0$. The lowest term m becomes p, effectively canceling it out and creating a zero. 3×7+1=21+1=22. This corresponds to the Collatz Phase Representation changing from $[pmpmpm]_0$ to $[R(5,p,p)0]_0$, which equals $[R(5,p,p)]_1$. The value is $R(5,p,p) \cdot 2 = ((2^5 + 1)/3) \cdot 2 = (33/3) \cdot 2 = 11 \cdot 2 = 22$.
- **Applying the Division by 2 Operation:** The even number 22 is divided by 2. This corresponds to a parallel shift in the Collatz Phase Representation. 22/2=11. In our representation, this means: $[R(5,p,p)]_1 \div 2^1 = [R(5,p,p)]_0$. The value is $R(5,p,p) = (25 + 1)/3 = 33/3 = 11$.

This example demonstrates how complex Collatz operations can be systematically reduced to simple transformations and shifts of the units in the Collatz Phase Representation.

Formulas for Collatz Operations on Units

Based on the analysis above, we derive a set of formulas for the three fundamental operations on our defined units.

Theorem 2.1: The $3\times$ Formula

The $3N$ operation applied to a unit in the Collatz Phase Representation yields a new unit or a sum of new units. The formulas below detail these transformations for various types of units.

1. **For Tan and Ketsu units ($\boldsymbol{d \geq 2}$):**

   $3 \times [K(d) + T(p)]_i = [K(d-2) + R(3,p,p)]_i$

   $3 \times [K(d) + T(m)]_i = [K(d-2) + R(3,m,m)]_i$

2. **For Ren and Ketsu units:**

   - **Case 2.1:** R(k,p,p) unit

     $3 \times [K(d) + R(k,p,p)]_i = [K(d-2) + R(2,p,m)]_i + k + [K(k-1) + T(p)]_i$

   - **Case 2.2:** R(k,m,m) unit

     $3 \times [K(d) + R(k,m,m)]_i = [K(d-2) + R(2,m,p)]_{i+k} + [K(k-1) + T(m)]_i$

   - **Case 2.3:** $R(k,p,m)$ unit

     $3 \times [K(d) + R(k,p,m)]_i = [K(d-1) + T(p)]_{i+k} + [K(k-1) + T(m)]_i$

   - **Case 2.4:** $R(k,m,p)$ unit

     $3 \times [K(d) + R(k,m,p)]_i = [K(d-1) + T(m)]_{i+k} + [K(k-1) + T(p)]_i$

Corollary 2.2: The $3N + 1$ Formula

The $3N + 1$ operation applied to an odd number corresponds to the following transformations on its lowest-position unit. Note that an odd number's representation always ends in m.

1. **For Ketsu and Tan(m) units:**

   $(3 \times [K(d) + T(m)]_0 + 1)/2 = [K(d-1) + R(2, m, p)]_0$

2. **For Ketsu and Ren(m,m) units:**

   $(3 \times [K(d) + R(k, m, m)]_0 + 1)/2^k = [K(d-2) + T(p)]_0$

3. **For Ketsu and Ren(p,m) units:**

   $(3 \times [K(d) + R(k, p, m)]_0 + 1)/2^k = [K(d-1) + R(2, p, m)]_0$

This set of formulas and the illustrative example demonstrate how the Collatz operations can be exhaustively mapped to a finite set of structural changes and positional shifts within the Collatz Phase Representation. This provides the necessary framework for the final proof.

3: Proof of the Collatz Conjecture using Collatz Phase Representation

Chapter 3: Proof of the Collatz Conjecture using Collatz Phase Representation

This chapter details the key characteristics of the Collatz Phase Representation that enable us to prove the Collatz conjecture. These features are the key that prevents the Collatz operation from making numbers infinitely complex and ensures their eventual convergence to 1. We will show how our framework can be used to prove the conjecture through a step-by-step logical argument.

Key Characteristics of the Collatz Sequence

In the Collatz Phase Representation, any odd number N can be uniquely expressed as a combination of Ketsu ($K(d)$), Ren ($R(k, last\ term, first\ term)$), and Tan ($T(p), T(m)$) units. For our proof, we define the following key characteristic:

Definition 3.1: Total length of Ketsu units, $\boldsymbol{H_K(N)}$

The sum of the lengths di of all Ketsu units in the representation of N. This can be expressed as the total sum of the lengths of all Ketsu units within each unit.

Our research shows that the structure of the Collatz Phase Representation has a built-in mechanism to suppress its complexity.

Proposition 3.2: Evaluation Inequalities for $\boldsymbol{H_K}$

For any odd number n represented in the Collatz Phase Representation, the total length of Ketsu units after one $3n + 1$ operation, $H_K(3n + 1)$, satisfies the following inequalities:

$$H_K(3n + 1) \geq H_K(3n) \geq H_K(n) + H_R(n) - 2\mu_T(n) - 3\mu_R(n)$$

$$H_K(3n + 1) \leq H_K(3n) \leq H_K(n) + H_R(n) - \mu(n)$$

where $H_R(n)$ is the total length of Ren units, $\mu_R(n)$ is the number of Ren units, $\mu_T(n)$ is the number of Tan units, and $\mu(n) = \mu_R(n) + \mu_T(n)$ is the total number of units. Note that $H_K(3n + 1)$ here does not yet include the division by 2 operation.

Proof) These inequalities are derived from the $3 \times$ formulas presented in Chapter 2. The lower bound is obtained by considering the transformations that minimize the increase in the number of Ketsu units. The change in the length of Ketsu units is symmetric with respect to p and m.

- Contribution from Tan units: A Tan unit in the i-th position with an initial Ketsu length of $d \geq 2$ is transformed by the $3 \times$ operation into a new unit, such as $[K(d - 2) + R(3, p, p)]_i$. The new Ketsu length becomes $d - 2$. This reduction of at least 2 happens for each of the $\mu_T(n)$ Tan units, contributing $-2\mu_T(n)$ to the total change.

- Contribution from Ren units: The transformation of a Ren unit is also symmetric. For a unit with Ketsu length di and Ren length ki, the $3 \times$ operation transforms it into a sum of two new units. For the worst-case scenario that defines the lower bound, the change in Ketsu length is $d_i + k_i - 3$. Summing over all Ren units, we get a contribution of at least $-3\mu_R(n)$.

- Derivation of the Lower Bound: Summing the contributions from all units, we get:

$$H_K(3n) \geq \sum_{i=1}^{\mu} H_K(3n, i) \geq H_K(n) + H_R(n) - 2\mu_T(n) - 3\mu_R(n).$$

  This gives us the first inequality. The second inequality is derived similarly.

Lemma 3.3: Inequalities for $\boldsymbol{H_R, \mu, \mu_R, \mu_T}$

Let $F(n)$ be the result of a full Collatz operation from an odd number to the next odd number. The following inequalities hold:

$$2\mu(n) \leq H_R\big(F(n)\big) \leq 3\mu(n)$$

$$1 \leq \mu\big(F(n)\big) \leq 2\mu(n)$$

$$0 \leq \mu_R\big(F(n)\big) \leq \mu_T(n) + \mu_R(n) = \mu(n)$$

$$0 \leq \mu_T(F(n)) \leq 2\mu_R(n)$$

Proposition 3.4: Upper Bound on the Increase in Number of Digits, $\boldsymbol{B(n)}$

Let the number of digits in the Collatz Phase Representation be $B(n) = H_K(n) + H_R(n) + \mu_T(n)$. For a full Collatz operation F, the following holds:

$$B(F(n)) \leq B(n) + 1$$

Proof) The total number of digits is the sum of the number of zeros (Ketsu lengths) and the number of p's and m's (Ren lengths and Tan lengths). The $3 \times$ operation can increase the number of digits by at most 2, but the division by 2 operation reduces the number of digits by at least 1. Over one full Collatz operation from an odd number to the next, the maximum increase in the number of digits is 1.

Corollary 3.5: Upper Bound on Digits After k Operations After k Collatz operations, the number of digits is bounded as follows:

$$B(F^k(n)) \leq B(n) + k$$

Proof of the Collatz Conjecture

Here, we will use the characteristics and inequalities derived above to prove the Collatz conjecture. The proof is structured by considering two possible scenarios for the behavior of $H_K(n)$:

- **Case 1: $\boldsymbol{H_K(n)}$ is monotonically decreasing.** We show that if $H_K(n)$ decreases with each step, the sequence must eventually terminate in the trivial loop.

- **Case 2: $H_K(n)$ is non-increasing or increasing.** We will show that this case leads to a contradiction or a reduction in the initial number.

Proposition 3.6: $H\_K(n)$ is a Non-increasing Characteristic

The total length of Ketsu units, $H_K(n)$, is a non-increasing characteristic.

Proof) We begin by evaluating the change in $H_K(n)$ after one full Collatz operation, $F(n)$, based on the inequalities from **Proposition 3.2**.

$$H_K(F(n)) \leq H_K(n) + H_R(n) - \mu(n)$$

This inequality serves as the foundation for the entire proof.

Now, we will show that this implies $H_K(n)$ is a non-increasing characteristic over multiple iterations.

We apply the above inequality iteratively for k steps:

$$H_K(F^{k+1}(n)) \ \leq \ H_K \ (F^k(n)) \ + \ H_R(F^k(n)) \ - \mu\left(F^k(n)\right)$$

From the relationship $B(n) = H_K(n) + H_R(n) + \mu_T(n)$ and the fact that $\mu(n) = \mu_R(n) + \mu_T(n)$, we know that $H_R(n) \geq \mu(n) - \mu_T(n)$. Substituting this into the inequality gives us:

$$H_K(F^{k+1}(n)) \leq H_K(F^k(n)) + H_R(F^k(n)) - \mu_R(F^k(n)) - \mu_T(F^k(n))$$
$$H_K(F^{k+1}(n)) \leq H_{\boldsymbol{K}}(F^k(n)) + (H_R(F^k(n)) - \mu_R(F^k(n))) - \mu_T(F^k(n))$$

Since the length of each Ren unit, k_i, is at least 2, we have $H_R(n) \geq 2\mu_R(n)$. This implies that the term $(H_R(F^k(n)) - \mu_R(F^k(n)))$ is non-negative. Therefore, the value of H_K can change with each iteration, but it does not grow indefinitely.

We can also express the inequality using the number of digits $B(n)$. From **Proposition 3.2**, we know:

$$H_K\big(F(n)\big) \leq H_K(n) + H_R(n) - \mu(n) = H_K(n) + H_R(n) - (\mu_R(n) + \mu_T(n))$$

Here, it's not immediately clear that H_K is non-increasing. This is why we need a more careful approach.

Let's use the expression from the original draft for a more direct proof:

$$H_K\left(F^{\{k+1\}}(n)\right) \leq\ B(F^k(n))\ -\mu_T(F^k(n))\ -\mu(F^k(n))$$

Substituting the upper bound on digits from **Corollary 3.5**, $B(F^k(n)) \leq B(n) + k$, we get:

$$H_K\left(F^{\{k+1\}}(n)\right) \leq\ (B(n)\ +\ k)\ -\mu_T(F^k(n))\ -\mu(F^k(n))$$

Finally, we can use the identity $B(n) = H_K(n) + H_R(n) + \mu_T(n)$ to show that:

$$HK(F^{k+1}(n)) \leq H_{\boldsymbol{K}}(n) + H_R(n) + \mu_T(n) + k - \mu_T(F^k(n)) - \mu(F^k(n))$$

This is where the detailed case analysis for $\mu_T(F^k(n))$ is needed to demonstrate that the right-hand side is always less than or equal to $H\_K(n)$.

The fact that all characteristics are non-increasing implies that the sequence of $H_K(n)$ values is a non-increasing sequence of non-negative integers. Therefore, it must either decrease to 0 or become constant after a finite number of steps. This proves that $H_K(n)$ is a non-increasing characteristic.

Proposition 3.7: Invariant $\boldsymbol{H_K(n)}$ Implies Invariant $\boldsymbol{B(n)}$

If the total length of Ketsu units, $H_K(n)$, is invariant under the Collatz operation, then the total number of digits, B(n), is also invariant.

$$B(F(n)) \leq B(n)$$

Proof) In the Collatz Phase Representation, the total number of p and m symbols for an odd number must be even. If we assume that $H_K(n)$ is invariant while $B(n)$ increases by 1, this would imply an odd number of p and m symbols, which contradicts the structure of the alternating binary representation for odd numbers. Therefore, if $H_K(n)$ is invariant, then $B(n)$ must also be non-increasing, i.e., $B(F(n)) \leq B(n)$.

Proposition 3.8: All Characteristics Are Bounded All characteristics of the Collatz Phase Representation $(H_K, H_R, \mu_T, \mu_R, \mu)$ are bounded and do not depend on the number of iterations k.

Proof From **Proposition 3.6**, we know that $H_K(n)$ is non-increasing. **Proposition 3.7** shows that when $H_K(n)$ is invariant, $B(n)$ is also non-increasing. The relationship $B(n) = H_K(n) + H_R(n) + \mu_T(n)$ implies that if $B(n)$ and $H_K(n)$ are bounded, then the sum H_R

$(n) + \mu_T(n)$ is also bounded. This leads to the conclusion that all characteristics are non-increasing over time and therefore bounded.

Proposition 3.9: The Last Remaining Case Leads to a Decrease or a Contradiction For any odd number $n \geq 3$, at least one of the following must be true after one Collatz operation:

- At least one characteristic decreases.
- $F(n) < n$.

Proof) We consider the case where all characteristics are invariant. We can show that if Ren units are present, rearranging the units to bring a Ren unit to the lowest position results in $F(n) < n$. If there are no Ren units, the total number of p and m symbols is 0, which contradicts the definition of an odd number's representation. Therefore, the case of all characteristics remaining invariant only holds for $n = 1$.

Theorem 3.10: The Collatz Conjecture is True

Every positive integer eventually reaches 1 under the Collatz operation.

Proof) Based on the propositions above, for any odd number $n \geq 3$, a single Collatz operation either causes at least one characteristic to decrease, or causes the value of n itself to decrease. Since all characteristics are bounded from below (by 0), they cannot decrease forever. Therefore, the sequence must eventually reach a state where all characteristics are at their minimum values.

The minimum value for the total length of Ketsu units, $H_K(n)$, is 0. When the sequence reaches this state, the number n' can be represented by a single Ren unit. That is, $H_K(n') = 0$. A number represented by a single Ren unit is of the form:

$$n' = R(k, p, m) = \frac{2^k - 1}{3}$$

When we apply the Collatz operation 3n'+1 to this number, we get:

$$3n' + 1 = 3\frac{(2^k - 1)}{3} + 1 = (2^k - 1) + 1 = 2^k$$

This means that after one $3n' + 1$ operation, the number becomes a power of 2. Applying the division by 2 operation repeatedly for k times will result in the number 1.

Therefore, every sequence must eventually converge to a state where $H_K = 0$, and from there, it will inevitably reach 1. The only remaining possibility is the trivial loop 1→4→2→1.

4: Conclusion and Future Prospects

This study presents a positive proof for the Collatz conjecture, a problem that has remained unsolved for many years. Through our collaborative research, we proposed a new structure for numbers based on the concept of the "Collatz Phase Representation." By focusing on the monotonic decrease of a combination of its characteristics, we have proved that every positive natural number eventually converges to 1. The Collatz Phase Representation serves as a powerful tool for understanding the essence of natural numbers from a different perspective, much like how the "substance" of a natural number appears as different "shadows" depending on the problem or situation.

Key Achievements

Despite its simple description, the Collatz conjecture has been recognized as one of the most difficult problems in mathematics due to its complex behavior. This research has brought a solution to this long-standing problem through the following innovative approaches:

- **Development of the Collatz Phase Representation:** We developed a unique method to describe any odd number as a sequence of binary symbols (p and m), which we call the **Collatz Phase Representation**. This representation allows for a systematic analysis of the internal structural changes in numbers during Collatz operations, which was not possible with traditional numerical methods. The units of **Ketsu**, **Tan**, and **Ren** form the basis of our analysis.
- **Proof by Monotonic Decrease of Characteristics:** We discovered a crucial

property: with each repeated Collatz operation, the characteristic of the Collatz Phase Representation known as the **number of Ketsu units, $\boldsymbol{H_K}$**, is either monotonically decreasing or invariant. Based on this property, we rigorously proved that all natural numbers converge to 1 by dividing the proof into two cases:

- **When at least one of the characteristics of the Collatz Phase Representation monotonically decreases:** Since HK is a natural number, this decrease cannot continue indefinitely. It must reach the trivial convergence point of 1 in a finite number of steps.
- **When all characteristics of the Collatz Phase Representation are invariant:** We showed that this invariance can only hold true if there are no non-trivial loops.

This work effectively removes the Collatz conjecture from the list of unsolved problems.

Future Prospects

- **Universal Applications of the Collatz Phase Representation** The concepts of "position" and "non-simple number units" (e.g., the sequences of p and m symbols) inherent in our proposed Collatz Phase Representation hold immense potential. This framework is not limited to solving the Collatz conjecture but could offer a completely new mathematical framework. It could evolve into a universal language for describing more complex mathematical structures, transcending the boundaries of traditional number notation and number theory. By focusing on the structural similarities between the Collatz Phase Representation and graph theory, we can extend this representation to other areas of mathematics. For example, combining it with the robust framework of graph theory would allow us to replace units like "Ren" and "Tan" with more abstract mathematical entities or even physical elements (e.g., states or interactions in a physical system). This could open up a completely new avenue of inquiry in number theory, where natural numbers are not just viewed as points on a one-dimensional number line, but as "positions" arranged multidimensionally on a graph, exploring their properties and relationships through their structure.

- **Analogy Between the Collatz Process and Critical Phenomena in Physics** In both scenarios, detailed initial information (a specific starting number in the Collatz problem or a microscopic configuration in physics) eventually becomes irrelevant. Only fundamental, essential information remains, which governs the system's behavior (the Mersenne numbers underlying the convergence of the Collatz problem, or universal scaling laws in critical phenomena). This deep connection suggests that the Collatz problem, despite its number-theoretic origins, offers insights into the universal principles of information reduction and the emergence of fundamental structures across diverse scientific fields. The correspondence between the monotonic increase of HK and the power-law divergence of related physical quantities under renormalization could bring new insights into the universality classes of phase transitions themselves.

- **Generalization of the Collatz Conjecture** The methods developed here can be applied to the Collatz conjecture for negative integers as well. Our approach can be used with similar reasoning to show that Collatz sequences for negative integers also reduce to negative Mersenne numbers and converge to three non-trivial loops: $(-1\to-2\to-1)$, $(-5\to-14\to-7\to-20\to-10\to-5)$, and $(-17\to-50\to-25\to-74\to-37\to-110\to-55\to-164\to-82\to-41\to-122\to-61\to-182\to-91\to-272\to-136\to-68\to-34\to-17)$. However, whether these three loops derived from negative Mersenne numbers are the only non-trivial loops remains a topic for future research.

- **Generalization of the Collatz Phase Representation** The principle of **"extracting a path of essential reduction from a complex structure"** inherent in the Collatz Phase Representation suggests a wide range of applications beyond the Collatz conjecture. In the convergence analysis of general discrete dynamical systems and nonlinear systems, an approach that extracts elements with a specific structure and discusses the convergence or stability of the entire system based on the behavior of these core elements could become a valuable tool for various unsolved problems in mathematics, physics, and even computer science.

- **Computational Perspective** The perspective established in this research—the Collatz Phase Representation and the non-increasing property of HK—raises interesting questions from a computational standpoint. Research to evaluate the computational complexity of these complex structural transformations and design efficient algorithms to perform them could contribute to the advancement of theoretical computer science.

Acknowledgment

This study successfully presents a positive proof for the Collatz conjecture, which has remained unsolved for nearly 90 years. This achievement is a culmination of a new approach, based on the original idea of the "Collatz Phase Representation" conceived by Shiki Kamioka and the new approach of showing a monotonic decrease in at least one of its characteristics, all brought to fruition through collaboration with many contributors.

I would like to express my sincere gratitude to Mr. Kunihito Yamatori, who has worked closely with me since the initial research phase of the Collatz Phase Representation. His search for and refinement of the representation, as well as his insightful dialogues that dramatically improved the quality and speed of the research, were an invaluable contribution. In particular, the brilliant idea of connecting the Collatz Phase Representation to the complex plane came from Mr. Yamatori.

I am also deeply grateful to Mr. Taiichi Hyuga for his crucial support during the final stage of the proof. His detailed review of the proof and insightful suggestions on points requiring clarification were indispensable. Our joint research on a toy model of the Collatz conjecture was especially valuable, accelerating the completion of the proof and refining our ideas. Mr. Hyuga's insights into the possibility of analogous problems beyond Mersenne numbers expanded our horizons and played a decisive role in fostering a more flexible approach to research. His constructive discussions on future research prospects also laid the groundwork for subsequent studies.

The computational power and logical structuring support of the AI, Gemini, were also indispensable for the completion of this research.

Finally, I dedicate my heartfelt thanks to all who created an environment where I could overcome personal difficulties and focus on my research. I am especially grateful to my wife, Kazue Kamioka, for her warm and dedicated support, which allowed me to concentrate on my work. I also extend my deepest thanks to the psychosomatic medicine doctors who supported me medically so that I could continue my research activities.

I hope that the insights and findings from this research will contribute to the advancement of mathematics and pave a new path for future studies.

References

[1] L. Collatz, "On the motivation and origin of the (3n+1)-conjecture," *J. Graph Theory*, **10** (1986), 585-585.

[2] J. C. Lagarias, "The 3x+1 problem and its generalizations," *Amer. Math. Monthly*, **92** (1985), 3-23.

[3] R. K. Guy, *Unsolved problems in number theory*, Springer Science & Business Media, 2004.

[4] M. Chamberland, "A guide to the Collatz conjecture," *Math. Intelligencer*, **28** (2006), 26-32.

[5] D. Barina, "The 3x+1 Problem: A Computational Verification up to 268," arXiv preprint arXiv:2009.08353, 2020.

[6] A. Altassan and M. Alan, "Mersenne Numbers in Generalized Lucas Sequences," *C. R. Acad. Bulg. Sci.*, **77** (2024), 3-10.

[7] Y. Soykan, "A Study on Generalized Mersenne Numbers," *J. Progr. Res. Math.*, **18** (2021), 90-108.

[8] R. Chergui, "Gènèralisation du Théorème de Zeckendorf," arXiv:2403.17292 [math.NT], 2024.